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Calibrations Modulo ν

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The theory of calibrations is extended to show that certain surfaces are area-minimizing modulo ν. For example, a complex algebraic variety in $\mathbf{C}^n$ of degree d is area-minimizing modulo ν for all $\nu \geqslant 2d$. © 1987 Academic Press, Inc.

Contents.

1. INTRODUCTION

The theory of calibrations has provided rich families of examples of m-dimensional area-minimizing surfaces. Unfortunately, the theory has not applied to the less restricted problem of minimizing area among unoriented surfaces or more general surfaces modulo ν. This paper introduces a theory of calibrations modulo ν. This theory alleviates the shortage of examples of area minimization modulo ν.

1.1. *Minimizing Area Modulo ν.* Let ν be an integer, $\nu \geqslant 2$. In the freer problem of minimizing area modulo ν, one allows the given boundary to be altered by extraneous pieces of multiplicity ν (and by limits of such). Figure 1.1(1) shows the standard length-minimizing curve bounded by eight given oriented points. Figure 1.1(2) shows the curve minimizing length modulo 2. Here four boundary points seem to have the wrong sign, the difference in each case being the point with multiplicity 2. In effect, "modulo 2" means "disregard orientation." Figure 1.1(3) shows the curve minimizing length modulo 3. The two extraneous boundary points have multiplicity 3 and hence do not count. Figure 1.1(4) shows curves minimizing length modulo 4, 5, 6, 7. However, for $\nu \geqslant 8$, the original curve of Fig. 1.1(1) minimizes length modulo ν.

A casual description of the theory of area minimization modulo ν appears in the introduction to [M5], and careful definitions in [F, 4.2.26]. Applications include modeling certain soap films and proving results in the standard theory of oriented surfaces [W2; M3, proof of Proposition 4.1]).

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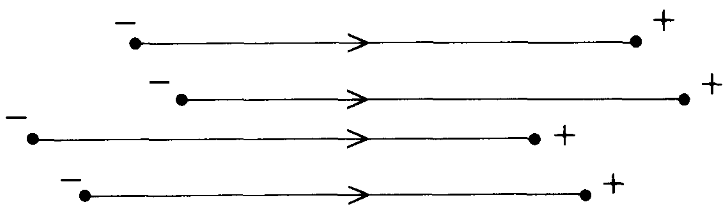

FIG. 1.1(1). A length-minimizing curve.

1.2. *Calibrations* [HL]. For positive integers $m \leqslant n$, let $G(m, \mathbf{R}^n)$ denote the Grassmannian of oriented, m-dimensional planes through the origin in $\mathbf{R}^n$. The Grassmannian $G(m, \mathbf{R}^n)$ may be viewed as a submanifold of the exterior algebra $\Lambda^m \mathbf{R}^n$ by indentifying the m-plane of oriented orthonormal basis $e_1, ..., e_m$ with the m-vector $e_1 \wedge \cdots \wedge e_m$.

A (standard, parallel) *calibration* in $\mathbf{R}^n$ is a constant-coefficient differential m-form $\phi \in \bigwedge^m \mathbf{R}^{n*}$ with *comass*

$$\|\phi\|^* \equiv \max\{\langle \xi, \phi \rangle : \phi \in G(m, \mathbf{R}^n)\} = 1.$$

Its set of maximum points

$$G(\phi) = \{\xi \in G(m, \mathbf{R}^n) : \langle \xi, \phi \rangle = 1\}$$

is called the *face* of the Grassmannian exposed by ϕ. The value of calibrations lies in the fact that *an m-dimensional surface, with tangent planes lying in a single face of the Grassmannian, is automatically area-minimizing* (among rectifiable currents with the same boundary).

1.3. *Calibrations Modulo* ν. We call a calibration ϕ a *calibration modulo* ν if ϕ can be expressed as a sum of simple covectors $\phi = \sum \phi_j$ such that the comass

$$\left\| \sum \sigma_j \phi_j \right\|^* \leqslant \|\phi\|^* = 1$$

FIG. 1.1(2). Length minimization modulo 2.

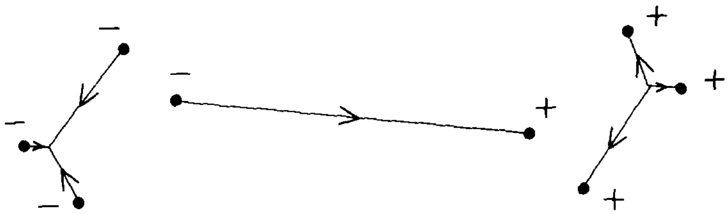

FIG. 1.1(3). Length minimization modulo 3.

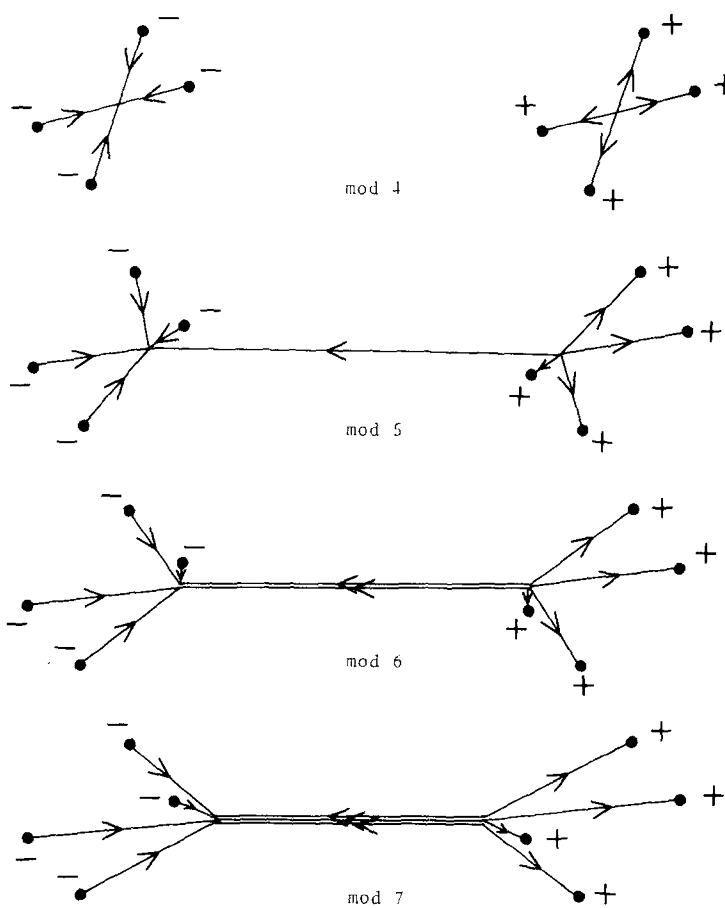

FIG. 1.1(4). Curves minimizing length modulo 4–7.

for all choices of signs $\sigma_j = \pm 1$. This stronger hypothesis on the calibration, together with a multiplicity bound on certain projections, implies that certain associated surfaces are area-minimizing modulo ν.

THEOREM (2.2). *Let ϕ be a calibration modulo ν. Let T be an m-dimensional surface, with tangent planes all lying in the face $G(\phi)$. Let Π_j denote orthogonal projection onto the (m-dimensional) space of ϕ_j. If for all j the multiplicity of $\Pi_{j\#} T$ is at most $\nu/2$, then T is area-minimizing modulo ν.*

Note that the example of Fig. 1.1(1), with multiplicity of projection 4, is area-minimizing modulo ν precisely when $\nu \geqslant 8$.

COROLLARY (Theorem 3.3). *Let T be a complex algebraic variety in $\mathbf{C}^n$ of degree d. Then T is area-minimizing modulo ν for all $\nu \geqslant 2d$.*

This corollary gives the first examples of area-minimizing surfaces modulo ν with branch points. The proof shows that powers of the Kähler form, suitably normalized, are calibrations modulo ν.

Indeed, Theorem 3.1 shows a large family of forms in $\Lambda^{2m}\mathbf{R}^{2n*}$ related to the Kähler form to be calibrations modulo ν. For $\Lambda^4\mathbf{R}^{8*}$, these calibrations have already been classified into 13 types [DHM, Theorem 2.7]. Other calibrations modulo ν include certain "double forms" in all dimensions (Section 4) and certain "diagonal forms" in $\Lambda^3\mathbf{R}^{7*}$ (Theorem 5.1).

Unfortunately, the calibrations with the richest geometries of area-minimizing surfaces, such as the special Lagrangian and Cayley forms, do not qualify as calibrations modulo ν (Proposition 5.6).

1.4. *Area-Minimizing Sums of Planes.* One of the first questions about singularities in area-minimizing surfaces asks when a k-tuple of planes through the origin is area-minimizing. See "Open problems in geometric measure theory," No. 5.8, [B′]. The theory of calibrations modulo ν gives some partial answers. The following theorem shows that the result for standard, oriented, 2-dimensional surfaces [M7, Corollary 4] also holds modulo ν for $\nu \geqslant 2k$.

THEOREM (3.4). *Let $\nu \geqslant 2k$. A sum of k 2-dimensional planes through the origin in $\mathbf{R}^n$ is area-minimizing modulo ν if and only if the planes are all complex planes for some orthogonal complex structure on their span.*

Such singularities are apparently unstable. However, for dimensions greater than 2, the situation is different.

THEOREM (4.5). *For $n \geqslant 2m \geqslant 6$, there is an open set of pairs of oriented m-dimensional planes in $\mathbf{R}^n$ which are area-minimizing modulo ν for all $\nu \geqslant 4$.*

This theorem indicates the existence of stable isolated singularites in area-minimizing surfaces modulo ν.

1.5. *Previous Examples.* It is hard to show that a particular surface is area-minimizing modulo ν. The few previous examples for $\nu = 2$ are described in "Examples of unoriented area-minimizing surfaces" [M1]; these examples actually work for all $\nu \geqslant 2$. See Remark 2.4 for further discussion.

2. Calibrations Modulo ν

Theorem 2.2, the cornerstone of this paper, describes how calibrations can be used to prove that certain surfaces are area-minimizing modulo ν.

2.1. Definitions. A form $\phi \in \Lambda^m \mathbf{R}^{n*}$ is a *calibration modulo* ν if ϕ can be expressed as a sum $\phi = \sum \phi_j$ of simple covectors such that for all $\sigma_j = \pm 1$,

$$\left\| \sum \sigma_j \phi_j \right\|^* \leqslant \|\phi\|^* = 1.$$

This definition is independent of ν (we always assume $\nu \geqslant 2$). Let $G(m, \mathbf{R}^n) \subset \Lambda^m \mathbf{R}^n$ denote the Grassmannian of oriented m-planes through 0 in $\mathbf{R}^n$. Call

$$G(\phi) = \{\xi \in G(m, \mathbf{R}^n) \colon \phi(\xi) = 1\}$$

the *face* of the Grassmannian exposed by ϕ.

2.2. Theorem. *For $n \geqslant m \geqslant 1$, $\nu \geqslant 2$, suppose that the sum $\phi = \sum \phi_j \in \Lambda^m \mathbf{R}^{n*}$ exhibits ϕ as a calibration modulo ν. Let T be a rectifiable current such that almost every oriented tangent plane lies in the face $G(\phi)$. Furthermore, letting Π_j denote orthogonal projection onto the (m-dimensional) space of ϕ_j, suppose that the essential supremum of the multiplicity of $\Pi_{j\#} T$ is at most $\nu/2$. Then T is area-minimizing modulo ν.*

Remarks. Taking $\phi = e_1^* \wedge \cdots \wedge e_m^*$ shows that for $k \leqslant \nu/2$, the union of k nearly coincident, unit, m-dimensional discs, parallel to the $e_1 \wedge \cdots \wedge e_m$ plane, is area-minimizing modulo ν. Since for $k > \nu/2$ such a union is not area-minimizing modulo ν (cf. Fig. 2.2(1)), this example shows that the multiplicity bound is sharp.

It is not true that every area-minimizing rectifiable current T is area-minimizing modulo ν for some ν. For example, consider the following countable collection of horizontal discs centered on the unit interval of the z-axis in $\mathbf{R}^3$. For each positive integer n, include n nearly coincident discs of

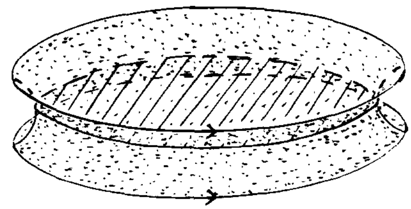

FIG. 2.2(1). $k = 2$ parallel discs are not area-minimizing modulo $\nu = 3 < 2k$.

radius $1/n \cdot 1/2^n$, centered between $(0, 0, 1/n)$ and $(0, 0, 1/(n+1))$. Then T is an area-minimizing rectifiable current, but T is not area-minimizing modulo n for any $n \geqslant 2$. Note that the total area $\mathbf{M}(T) = \sum n \cdot \pi(1/n \cdot 1/2^n)^2 \leqslant \pi \sum (1/4^n) = \pi/3 < \infty$, and $\mathbf{M}(\partial T) = \sum n \cdot 2\pi(1/n \cdot 1/2^n) = 2\pi < \infty$.

Having illustrated the necessity of a multiplicity bound, we now point out the necessity of the calibration. Even if T is area-minimizing and the multiplicity of its projection to every axis m-plane is less than or equal to $\nu/2$, T need not be area-minimizing modulo ν. See Fig. 2.2(2). However, I do not know an example of a classically calibrated surface T which satisfies the multiplicity bound but is not area-minimizing modulo ν.

This theorem shows that certain oriented surfaces minimize area in the larger class of unoriented surfaces (orientable or nonorientable) with the same boundary (mod 2). One might attempt to use this theorem to show that some *nonorientable* surface T with reasonable boundary is area-minimizing by first orienting T by introducing boundary of multiplicity 2. However, such an approach cannot work. Recall that T must be regular almost everywhere [F2]. Since $\langle \xi, \phi \rangle = -\langle -\xi, \phi \rangle$, at each regular point of T there is at most 1 oriented tangent plane $\vec{T}$ such that $\langle \vec{T}, \phi \rangle = 1$, and $\vec{T}$ varies continuously. Therefore if $\langle \vec{T}, \phi \rangle = 1$ almost everywhere, $\vec{T}$ gives an orientation of T.

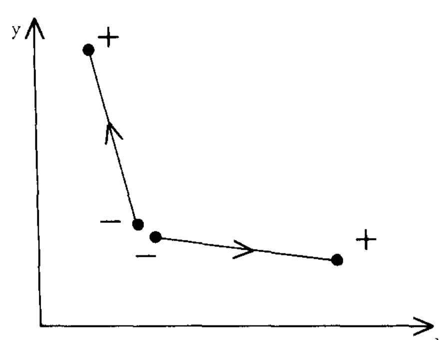

FIG. 2.2(2). Length-minimizing, but not modulo 2.

2.3. LEMMA. *Let $\phi = \sum \phi_j$ exhibit ϕ as a calibration modulo ν. Then for all j, for all ξ in the face $G(\phi)$, $\langle \xi, \phi_j \rangle \geq 0$.*

Proof. By the definition of a calibration modulo ν, $\|\phi - 2\phi_j\|^* \leq 1$. Therefore

$$\langle \xi, \phi_j \rangle = \tfrac{1}{2}(\langle \xi, \phi \rangle - \langle \xi, \phi - 2\phi_j \rangle) \geq 0.$$

Proof of the Theorem. First, since the tangent planes to T lie in the face $G(\phi)$, $\mathbf{M}(T) = T(\phi) = \sum T(\phi_j)$. By Lemma 2.3, for each j, $T(\phi_j) = |\phi_j| \mathbf{M}(\Pi_{j\#} T)$. Therefore

$$\mathbf{M}(T) = \sum |\phi_j| \, \mathbf{M}(\Pi_{j\#} T).$$

Second, let S be any rectifiable current with $\partial S \equiv \partial T \pmod{\nu}$. Of course, $\partial(\Pi_{j\#} S) \equiv \partial(\Pi_{j\#} T) \pmod{\nu}$. Since by hypothesis the multiplicity of $\Pi_{j\#} T \leq \nu/2$,

$$\mathbf{M}(\Pi_{j\#} T) \leq \mathbf{M}(\Pi_{j\#} S).$$

Third,

$$\sum |\phi_j| \, \mathbf{M}(\Pi_{j\#} S) \leq \sum \int |\langle \vec{S}, \phi_j \rangle| \, d\|S\| = \int \left\langle \vec{S}, \sum \sigma_j \phi_j \right\rangle d\|S\|,$$

where $\sigma_j = \operatorname{sign}\langle \vec{S}, \phi_j \rangle$. Finally,

$$\int \left\langle \vec{S}, \sum \sigma_j \phi_j \right\rangle d\|S\| \leq \mathbf{M}(S),$$

because $\phi = \sum \phi_j$ is a calibration modulo ν. Combining our results yields $\mathbf{M}(T) \leq \mathbf{M}(S)$, as desired.

2.4. *Remark.* Previous examples of area-minimizing surfaces modulo $\nu = 2$ appeared in [M1]. They generalize to all $\nu \geq 2$. The generalizations of Theorems 2 and 4 of [M1] appear as Corollaries 2.5 and 2.6 of [M5]. Theorems 3 and 5 of [M1] and their proofs remain valid modulo ν.

In dimension 1, the union of ν unit vectors from the origin in $\mathbf{R}^n$ with vector sum 0 is length-minimizing modulo ν (see [Ab]). Also, the Cartesian product with an interval of a surface which is area-minimizing modulo ν is area-minimizing modulo ν. [M4, Lemma 2.1].

For $\nu = 2$ or 3, it is an open question whether the Cartesian product of two surfaces which are area-minimizing modulo ν is itself area-minimizing modulo ν. See "Open problems in geometric measure theory," No. 3.7, [B′]. For $\nu \geq 4$, the answer is negative. For example, the Cartesian product of $[\nu/2]$ nearly coincident parallel planes with itself yields $[\nu/2]^2$ nearly parallel planes, which is not area-minimizing modulo ν, since $[\nu/2]^2 > \nu/2$. Moreover, the example of L. C. Young [Y], as proved in [W1], actually gives for all $\nu \geqslant 4$ a 2-dimensional surface T in $\mathbf{R}^4$ such that T minimizes area modulo ν but $2T$ does not. Consequently, the Cartesian product of T with two long, nearly coincident intervals does not minimize area modulo ν.

3. Kähler and Related Forms

The following theorem provides the most important examples of calibrations modulo ν. Applications to area minimization involve complex algebraic varieties (Theorems 3.2 and 3.3) and sums of 2-dimensional planes (Theorem 3.4).

3.1. Theorem. *For $1 \leqslant m \leqslant n-1$, view $\mathbf{R}^{2n} \cong \mathbf{C}^n$, with real orthonormal basis $e_1, ie_1, \ldots, e_n, ie_n$. Let $\omega_j = e_j^* \wedge (ie_j)^*$. For multiindex J with $1 \leqslant J_1 \leqslant \cdots \leqslant J_m \leqslant n$, let $\omega_J = \omega_{J_1} \wedge \cdots \wedge \omega_{J_m} \in \Lambda^{2m}\mathbf{R}^{2n*}$ and let*

$$\phi = \sum a_J \omega_J$$

with $a_J \in \{-1, 0, 1\}$. Then ϕ is a calibration modulo ν. In particular, powers of the Kähler form $\omega = \sum \omega_j$, suitably normalized, are calibrations modulo ν.

Proof. Theorem 2.2 of [DHM] says that such a form ϕ is a calibration. Since switching signs on the a_J gives another ϕ of the same form, ϕ is a calibration modulo ν.

Remark. For $m = 1$, each calibration provided by Theorem 3.1 is just the Kähler form on some $\mathbf{C}^k \subset \mathbf{C}^n$ (modulo trivial sign changes). The dual case $m = n-1$ also provides nothing beyond familiar Kähler geometry. For $p = 2$, $m = 4$, there are already 13 types of such calibrations, as described by [DHM, Theorem 2.7].

3.2. Theorem. *Let T be a complex algebraic hypersurface in $\mathbf{C}^n$, defined by a polynomial of degree at most d in each variable. Then T is area-minimizing modulo ν for all $\nu \geqslant 2d$.*

Remark. For example, $\{(z, w) \in \mathbf{C}^2 : zw = 1\}$ is area-minimizing mod 2. R. Hardt and F.-H. Lin [H Lin] have proved that every orientable 2-dimensional area-minimizing flat chain modulo 2 without boundary in $\mathbf{R}^4$ is orthogonally equivalent to the plane $\{w = 0\}$, orthogonal planes $\{zw = 0\}$, or $\{zw = c > 0\}$.

Proof of the Theorem. By Theorem 3.1, $\Omega = \omega^{m-1}/(m-1)!$, where ω is the Kähler form $\omega = \sum \omega_j$, is a calibration mod v. In particular, $\Omega = \sum \phi_j$, where $\phi_j = \omega_j \,\lrcorner\, \omega_1 \wedge \cdots \wedge \omega_m$. The hypothesis on the degree guarantees that the multiplicity of the projection of T on the space of ϕ_j, which is span $\{e_j, ie_j\}^\perp$, is at most $v/2$.

By Wirtinger's Inequality [*F*, 1.8.2], Ω is 1 on complex hyperplanes, and hence $\mathbf{M}(T_0) = T_0(\Omega)$, for any compact portion T_0 of T. Therefore by Theorem 2.2, T is area-minimizing mod v.

3.3. THEOREM. *Let T be a complex algebraic variety in $\mathbf{C}^n$ of degree d. Then T is area-minimizing modulo v for $v \geq 2d$.*

Remark. If T is defined by complex polynomials, it follows from Bezout's theorem [H, Theorem I.7.7] that the degree of T is at most the product of the degrees of the defining polynomials.

Proof. Suppose T is an m-dimensional complex variety in $\mathbf{C}^n$. By Theorem 3.1, the form $\Omega = \omega^m/m!$ is a calibration modulo v. By Wirtinger's Inequality [F, 1.8.2], Ω is 1 on complex planes, and hence $\mathbf{M}(T_0) = T_0(\phi)$ for any compact portion T_0 of T. The hypothesis on the degree means that almost all fibers of a projection map intersect T in at most $v/2$ points (cf. [H, Sect. I.7]). Therefore by Theorem 2.2, T is area-minimizing mod v.

The following theorem shows that the criterion of [M7, Corollary 4] for a union of 2-dimensional planes to be area-minimizing also holds modulo v for $v \geq 4$.

3.4. THEOREM. *Let $v \geq 2k$. A sum of k 2-dimensional planes in $\mathbf{R}^n$ is area-minimizing modulo v if and only if the planes are all complex planes for some orthogonal complex structure on their span.*

Proof. If a sum of planes is area-minimizing mod v, then of course it is area-minimizing among rectifiable currents. Consequently by [M7, Corollary 4] the planes are all complex planes for some orthogonal complex structure on their span.

Conversely, if $v \geq 2k$, a sum of k complex planes is area-minimizing mod v by Theorem 3.3.

Remarks. For $v = 2$, a sum of k 2-dimensional planes in $\mathbf{R}^n$ is area-minimizing modulo 2 if and only if the planes are orthogonal [M7, Corollary 7]. For $3 \leq v < 2k$, {sums of k orthogonal planes} $\subset$ {area-minimizing sums of k planes} $\subset$ {sums of k complex planes}. The first inclusion follows from [M1, Theorem 5] and Remark 2.4, the second from [M7, Corollary 4]. Consideration of k nearly coincident complex planes shows the second inclusion to be proper. The properness of the first inclusion is only conjectured, even in the case $v = 3$, $k = 2$.

4. Double Forms and Pairs of Planes

A basic question in the study of singularities asks which pairs of planes are area-minimizing. See "Open problems in geometric measure theory," No. 5.8, [B′]. To date, certain pairs of planes in certain dimensions have been proved area-minimizing among rectifiable currents by exhibiting "double forms," i.e., calibrations which attain the value 1 on precisely the two given planes [M6, Introduction and Theorem 3; M2, 1.2 and Theorem 4.13; DH, Theorem 8]. Some of those forms can be shown to qualify as calibrations modulo ν. In particular, Theorem 4.3 identifies certain pairs of 3-planes as area-minimizing modulo ν for all $\nu \geqslant 4$. Theorem 4.5 provides an open set of pairs of k-planes which are area-minimizing modulo ν for all $k \geqslant 3$, $\nu \geqslant 4$.

The following proposition provides an important class of calibrations modulo ν.

4.1. Proposition. *For $m \geqslant 1$, $\mathbf{R}^{2m} \cong \mathbf{C}^m$, let ϕ be a calibration in $\Lambda^m \mathbf{R}^{2m*}$ of torus form*

$$\phi(\lambda) = \sum \lambda_J \begin{Bmatrix} e_1^* \\ \text{or} \\ (ie_1)^* \end{Bmatrix} \wedge \cdots \wedge \begin{Bmatrix} e_m^* \\ \text{or} \\ (ie_m)^* \end{Bmatrix}.$$

If $\lambda_J \geqslant 0$ for all indices J, then ϕ is a calibration modulo ν.

Proof. We are given that $\|\phi(\lambda)\|^* = 1$ and we must show that for any choice of signs $\sigma_J = \pm 1$, $\|\phi((\sigma_J \lambda_J))\|^* \leqslant 1$. By the Torus Lemma ([M6, Lemma 4] or [DHM, Theorem 4.2]), $\phi((\sigma_J \lambda_J))$ attains its maximum value on an m-vector of the form

$$\xi(\theta) = (\cos\theta_1 e_1 + \sin\theta_1 ie_1) \wedge \cdots \wedge (\cos\theta_m e_m + \sin\theta_m ie_m).$$

Therefore

$$\begin{aligned} \|\phi((\sigma_J \lambda_J))\|^* &= \langle \xi(\theta), \phi((\sigma_J \lambda_J)) \rangle \\ &\leqslant \langle (|\cos\theta_1|\, e_1 + |\sin\theta_1|\, ie_1) \\ &\qquad \wedge \cdots \wedge (|\cos\theta_m|\, e_m + |\sin\theta_m|\, ie_m), \phi((\lambda_J)) \rangle \\ &\leqslant \|\phi(\lambda)\|^* = 1. \end{aligned}$$

4.2. *Remark* (Cf. [HL, Lemma 7.5] or [M2, Lemma 2.6]). View $\mathbf{R}^{2m} \cong \mathbf{C}^m$, with real orthonormal basis $e_1, ie_1, \ldots, e_m, ie_m$. Every pair of oriented m-planes in $\mathbf{R}^{2m}$ is orthogonally equivalent to a pair $\xi(0)$, $\xi(\gamma)$, where

$$\xi(\gamma)=e^{i\gamma_1}e_1\wedge\cdots\wedge e^{i\gamma_m}e_m,$$

$0\leqslant\gamma_1\leqslant\cdots\leqslant\gamma_{m-1}\leqslant\pi/2$, $\gamma_{m-1}\leqslant\gamma_m\leqslant\pi-\gamma_{m-1}$. The angles γ_j are unique and thus characterize the geometric relationship between the planes.

The following theorem of [K] provides a sufficient condition for a pair of 3-dimensional planes to be area-minimizing modulo v for all $v\geqslant 4$.

4.3. THEOREM. *Consider a pair of 3-dimensional planes in $\mathbf{R}^n$ with characterizing angles $0\leqslant\gamma_1\leqslant\gamma_2\leqslant\pi/2$, $\gamma_2\leqslant\gamma_3\leqslant\pi-\gamma_2$. If $1+\cos\gamma_3\geqslant\cos\gamma_1+\cos\gamma_2$, then the pair of planes is area-minimizing modulo v for all $v\geqslant 4$.*

Remarks. Our proof produces a suitable torus form with nonnegative coefficients, which is a calibration modulo v by Proposition 4.1. Consideration of nearly coincident planes shows that the theorem fails for $v=2$ or 3.

The first proof, in Kane [K], uses a comass formula of [HM1, Theorem 2.20]. Kane also exhibits calibrations modulo v by nonaxis decompositions and computer calculations.

Proof. We may assume that the two planes are $\xi(\gamma/2)=e^{i\gamma_1/2}e_1\wedge e^{i\gamma_1/2}e_2\wedge e^{i\gamma_3/2}e_3$ and $\xi(-\gamma/2)$. Note that

$$\begin{aligned}\cos\gamma_3&\geqslant\cos\gamma_1+\cos\gamma_2-1=\cos\gamma_1\cos\gamma_2-(1-\cos\gamma_1)(1-\cos\gamma_2)\\&\geqslant\cos\gamma_1\cos\gamma_2-\sin\gamma_1\sin\gamma_2=\cos(\gamma_1+\gamma_2),\end{aligned}$$

with equality only in the trivial case $\gamma=0$. Hence we may assume $\gamma_3<\gamma_1+\gamma_2$. Then [M6, Theorem 3] or [HM, Theorem 2.3] provides a calibration

$$\phi(\lambda)=\lambda_0e^*_{123}+\lambda_1e^*_{156}+\lambda_2e^*_{426}+\lambda_3e^*_{453}$$

with $\langle\xi(\pm\gamma/2),\phi\rangle=\|\phi\|^*=1$. If $c_j=\cos\gamma_j/2$ and $s_j=\sin\gamma_j/2$, then

$$\lambda_0=\frac{c_1^2+c_2^2+c_3^2-1}{2c_1c_2c_3},$$

$$\lambda_1=\frac{c_1^2-c_2^2-c_3^2+1}{2c_1s_2s_3},$$

$$\lambda_2=\frac{-c_1^2+c_2^2-c_3^2+1}{2s_1c_2s_3},$$

$$\lambda_3=\frac{-c_1^2-c_2^2+c_3^2+1}{2s_1s_2c_3}.$$

Since $0 \leqslant c_3 \leqslant c_2 \leqslant c_1 \leqslant 1$, therefore λ_0, λ_1, and λ_2 are nonnegative. The hypothesis that $1+\cos\gamma_3 \geqslant \cos\gamma_1 + \cos\gamma_2$ implies that $-\cos^2\gamma_1/2 - \cos^2\gamma_2/2 + \cos^2\gamma_3/2 + 1 \geqslant 0$ and λ_3 is nonnegative. Thence, by Proposition 4.1, $\phi(\lambda)$ is a calibration mod ν. Therefore by Theorem 2.2, the pair of planes is area-minimizing mod ν for all $\nu \geqslant 4$.

4.4. *Remarks.* For $\nu \geqslant 4$, the preceding theorem gives an open set of pairs of 3-dimensional planes which are area-minimizing modulo ν. Such evidence suggests the existence of singularities in area-minimizing flat chains modulo ν which are stable under general perturbations of the boundary. Theorem 4.5 deduces a similar result in higher dimensions with the help of the Implicit Function Theorem. (Cf. [M6, Introduction].)

4.5. THEOREM. *For $n \geqslant 2m \geqslant 6$, there is an open set of pairs of oriented m-dimensional planes in $\mathbf{R}^n$ which are area-minimizing modulo ν for all $\nu \geqslant 4$. Indeed, each such pair is the face of a calibration of torus form with nonnegative coefficients (in terms of some orthonormal basis for $\mathbf{R}^n$). For m even, it suffices to take the planes nearly orthogonal.*

Remark. The theorem remains conjectural for $\nu = 2$ or $\nu = 3$. Theorem 2.2 does not apply because the multiplicity of projection $2 > \nu/2$. For $m < 3$, the theorem is false (see Theorem 3.4).

Proof. By Proposition 4.1 and Theorem 2.2, it suffices to produce the asserted calibration. Since any pair of m-planes is contained in some $2m$-dimensional subspace of $\mathbf{R}^n$, it suffices to consider the case $n = 2m \geqslant 6$. View $\mathbf{R}^{2m} \cong \mathbf{C}^m$, with real orthonormal basis $e_1, ie_1, \dots, e_m, ie_m$. For parameter $\lambda = (\lambda_1, \dots, \lambda_m) \in \mathbf{R}^m$, consider the m-form $\psi(\lambda)$ in $\Lambda^m \mathbf{R}^{2m*}$ given by

$$\psi(\lambda) = (e_1^* + \lambda_1 (ie_1)^*) \wedge \cdots \wedge (e_m^* + \lambda_m (ie_m)^*).$$

Let τ denote the orthogonal map switching each e_j and ie_j. Let $\phi(\lambda)$ be the torus form given by

$$\phi(\lambda) = \psi(\lambda) + \tau^{\#}\psi(\lambda).$$

First we will be interested in applying $\phi(\lambda)$ to m-vectors of the form

$$\xi(\theta) = e^{i\theta_1} e_1 \wedge \cdots \wedge e^{i\theta_m} e_m.$$

For $-\pi/4 \leqslant \theta_j < 3\pi/4$ $(1 \leqslant j \leqslant m-1)$, $-\pi/4 \leqslant \theta_m < 7\pi/4$, let

$$f_\lambda(\theta) = \langle \xi(\theta), \phi(\lambda) \rangle.$$

The restriction on θ does not restrict the values of $\xi(\theta)$, because changing one θ_j by 2π or two by π does not change $\xi(\theta)$. The function

$$\begin{aligned} f_0(\theta) &= \cos\theta_1 \cdots \cos\theta_m + \sin\theta_1 \cdots \sin\theta_m \\ &\leqslant |\cos\theta_1 \cos\theta_2| + |\sin\theta_1 \sin\theta_2| \\ &= |\cos(\theta_1 \pm \theta_2)| \leqslant 1, \end{aligned}$$

with equality if and only if $\theta = 0 = (0, \ldots, 0)$ or $\theta = \pi/2 = (\pi/2, \ldots, \pi/2)$. Therefore the absolute maxima of f_0 are precisely 0 and $\pi/2$. Moreover,

$$\left.\frac{\partial^2 f_\lambda}{\partial\theta_j^2}\right|_{\theta=\lambda=0} = \left\langle \frac{\partial^2\xi}{\partial\theta_j^2}(0), \phi(0) \right\rangle = \langle -\xi(0), \phi(0) \rangle = -1,$$

while for $j \neq k$, $m \geqslant 3$,

$$\begin{aligned} \left.\frac{\partial^2 f_\lambda}{\partial\theta_j\,\partial\theta_k}\right|_{\theta=\lambda=0} &= \left\langle \frac{\partial^2\xi}{\partial\theta_j\,\partial\theta_k}(0), \phi(0) \right\rangle \\ &= \langle e_1 \wedge \cdots \wedge ie_j \wedge \cdots \wedge ie_k \wedge \cdots \wedge e_m, \\ &\quad\; e_1 \wedge \cdots \wedge e_m + (ie_1) \wedge \cdots \wedge (ie_m) \rangle \\ &= 0. \end{aligned}$$

In particular, Df_0 is negative definite at 0. Therefore, by the Implicit Function Theorem, there is a smooth map $\theta = g(\lambda)$ on a neighborhood U of $\lambda = 0$ such that f_λ has a critical point at $g(\lambda)$. Since $\tau^{\#}\phi(\lambda) = \phi(\lambda)$, f_λ has the symmetry $f_\lambda(\pi/2 - \theta) = f_\lambda(\theta)$. Therefore, by shrinking U if necessary, we may assume that for $\lambda \in U$, the absolute maxima of f_λ are precisely $g(\lambda)$ and $\pi/2 - g(\lambda)$. It follows by the Torus Lemma ([M6, Lemma 4] or [DHM, Theorem 4.2]) that the absolute maxima of $\phi(\lambda)$ as a function on the Grassmannian $G(m, \mathbf{R}^{2m})$ are precisely $\xi(g(\lambda))$ and $\xi(\pi/2 - g(\lambda))$.

Next we employ the Implicit Function Theorem to show that g is a local diffeomorphism. For $m \geqslant 3$,

$$\begin{aligned} \left.\frac{\partial}{\partial\lambda_j}\left(Df_\lambda(0)\left(\frac{\partial}{\partial\theta_k}\right)\right)\right|_{\lambda=0} &= \left\langle \frac{\partial\xi}{\partial\theta_k}(0), \frac{\partial\phi}{\partial\lambda_j}(0) \right\rangle \\ &= \langle e_1 \wedge \cdots \wedge ie_k \wedge \cdots \wedge e_m, \\ &\quad\; e_1^* \wedge \cdots \wedge (ie_j)^* \wedge \cdots \wedge e_m \\ &\quad\; + (ie_1)^* \wedge \cdots \wedge e_j^* \wedge \cdots \wedge (ie_m)^* \rangle \\ &= \delta_{jk} \end{aligned}$$

is the δ-function. In particular, $((\partial/\partial\lambda_j)\, Df_\lambda)|_{\theta=\lambda=0}$ has rank m. Therefore by shrinking U if necessary, we may assume that g is a smooth diffeomorphism of U onto a small neighborhood V of $\theta=0$. We now proceed by cases according to whether m is even or odd, which bears on whether ϕ has nonnegative coefficients.

Case m even. For m even, for all λ, momentarily replacing each basis vector ie_j by (sign λ_j) (ie_j) shows that $\phi(\lambda)$ has nonnegative coefficients for some choice of orthonormal basis. For all θ in the neighborhood V of 0, the pair $\{\xi(\theta), \xi(\pi/2-\theta)\}$ is the face of the form $\phi(g^{-1}(\theta))$. Rotating this pair by the angle θ_j in each $e_j \wedge ie_j$ plane shows that the pair $\xi(0)$, $\xi(\pi/2-2\theta)$ is the face of a suitable calibration. For $\theta \in \{\theta \in V: |\theta_m| \leqslant \theta_{m-1} \leqslant \cdots \leqslant \theta_1\}$, the characterizing angles of Remark 4.2 are given by $\gamma_j = \pi/2 - 2\theta_j$, and they fill out an open neighborhood of $(\pi/2, \ldots, \pi/2)$ in $\{0 \leqslant \gamma_1 \leqslant \cdots \leqslant \gamma_{m-1} \leqslant \pi/2,\ \gamma_{m-1} \leqslant \gamma_m \leqslant \pi - \gamma_{m-1}\}$. The proof is complete for m even.

Case m odd. Let V' be an open subset of $g\{\lambda \in U: \lambda_j \geqslant 0\} \subset V$. V' may well not contain 0, but for all $\theta \in V'$, the form $\phi(g^{-1}(\theta))$ has nonnegative coefficients and face $\{\xi(\theta), \xi(\pi/2-\theta)\}$. Rotating this pair by the angle θ_j in each $e_j \wedge ie_j$ plane shows that the pair $\xi(0)$, $\xi(\pi/2-2\theta)$ is the face of a suitable calibration. By shrinking V' and permuting the indices $1, 2, \ldots, m$, we may assume that for all $\theta \in V'$, $0 < |\theta_m| < |\theta_{m-1}| < \cdots < |\theta_1|$. Then the characterizing angles of Remark 4.2 are given by $\gamma_j = \pi/2 - 2|\theta_j|$ $(1 \leqslant j \leqslant m-1)$, $\gamma_m = \pi/2 - 2\sigma|\theta_m|$, where $\sigma = \prod_{j=1}^m (\text{sign}\,\theta_j) = \pm 1$. These angles fill out an open set and complete the proof.

The following theorem provides another method of constructing calibrations.

4.6. Theorem. *For $m \geqslant 1$, $n \geqslant k \geqslant 1$, consider $\mathbf{R}^{2m} \cong \mathbf{C}^m$ with orthonormal basis $e_1, ie_1, \ldots, e_m, ie_m$. Let $\phi \in \Lambda^m \mathbf{R}^{2m*}$, $\psi \in \Lambda^k \mathbf{R}^{n*}$ be calibrations. Suppose that ϕ is of torus form*

$$\phi = \sum a_J \left\{ \begin{matrix} e_1^* \\ \text{or} \\ ie_1^* \end{matrix} \right\} \wedge \cdots \wedge \left\{ \begin{matrix} e_m^* \\ \text{or} \\ ie_m^* \end{matrix} \right\}.$$

Then $\phi \wedge \psi \in \Lambda^{m+k} \mathbf{R}^{2m+n*}$ is a calibration, and $G(\phi \wedge \psi) = G(\phi) \wedge G(\psi)$.

Remark. Every $\phi \in \Lambda^3 \mathbf{R}^{6*}$ is of torus form for some choice of orthonormal basis for $\mathbf{R}^6$ [M2, Theorem 4.1], and the theorem applies. In particular, for $1 \leqslant k \leqslant n$, if $\psi \in \Lambda^k \mathbf{R}^{n*}$, the product $\phi \wedge \psi \in \Lambda^{3+k} \mathbf{R}^{6+n*}$ satisfies $\|\phi \wedge \psi\|^* = \|\phi\|^* \|\psi\|^*$. It follows that the Cartesian product of a 3-dimensional, area-minimizing normal current in $\mathbf{R}^6$ and a k-dimensional area-minimizing normal current in $\mathbf{R}^n$ is itself area-minimizing [M2, Sect. 5]. The question remains open in general dimensions. See "Open problems in geometric measure theory," No. 3.7, [B′].

This theorem adds large families of examples to the standard theory of calibrations.

Proof. We fix n and prove the theorem by induction on m and k. The result is easy for $m=1$ or $k=1$ (cf. [M2, proof of 4.2]).

For $m, k \geqslant 2$, let ξ be an $(m+k)$-plane in $G(m+k, \mathbf{R}^{2m+n})$ on which $\phi \wedge \psi$ attains its maximum. By Lemma 4.1 of [DHM], ξ has a factor $\eta \in G(m+k-2, \mathbf{R}^{2(m-1)} \times \mathbf{R}^n)$ such that for some unit vectors $v \in \operatorname{span}\{e_m, ie_m\}$, $w \in \mathbf{R}^{2(m-1)} \times \mathbf{R}^n$, perpendicular to η, $\phi \wedge \psi$ attains its maximum on $\eta \wedge w \wedge v$ (which may or may not be ξ). Hence $v \,\lrcorner\, \phi \wedge \psi$ attains its maximum on $\eta \wedge w$. Note that $v \,\lrcorner\, \phi \in \Lambda^{m-1}\mathbf{R}^{2(m-1)*}$ is of torus form. Therefore by induction, $\eta \wedge w \in G(v \,\lrcorner\, \phi) \wedge G(\psi)$. Since $k = \text{degree } \psi \geqslant 2$, $\eta \wedge w$ and hence ξ have a unit vector factor $u \in \mathbf{R}^n$. Hence $u \,\lrcorner\, \phi \wedge \psi = (-1)^m \ \phi \wedge (u \,\lrcorner\, \psi)$ attains its maximum on $\xi \llcorner u^*$. By induction, $\xi \llcorner u^* \in (-1)^m \ G(\phi) \wedge G(u \,\lrcorner\, \psi)$. Therefore, $\xi \in G(\phi) \wedge G(\psi)$. In particular, $\|\phi \wedge \psi\|^* = 1$.

4.7. *Quadruple Forms.* Applying Theorem 4.6 to a pair of double forms ϕ, ϕ' of Theorem 4.4 yields a calibration $\psi = \phi \wedge \phi' \in \Lambda^{2m}(\mathbf{R}^{2m} \times \mathbf{R}^{2m})^*$, and $G(\psi) = G(\phi) \wedge G(\phi')$ consists of precisely four planes. Since ϕ and ϕ' are torus forms with nonnegative coefficients, so is ψ. Therefore, by Proposition 4.1, ψ is a calibration modulo v. In particular, for $m \geqslant 3$, by Theorem 2.2, certain quadruples of $2m$-dimensional planes in $\mathbf{R}^{4m}$ are area-minimizing modulo v for all $v \geqslant 8$. Iterating this argument yields for all k, for $m \geqslant 3$, calibrations modulo v ϕ in $\Lambda^{km}\mathbf{R}^{2km*}$ with face $G(\phi)$ consisting of precisely 2^k km-dimensional planes in $\mathbf{R}^{2km}$.

For $m \geqslant 2$, $k \geqslant 3$, earlier examples of sums of k m-dimensional planes in $\mathbf{R}^{mk}$ which are area-minimizing modulo v for all $v \geqslant 2$ were provided by [M1, Theorem 5] (see Remark 2.4). For example, k orthogonal planes are area-minimizing modulo v for all $v \geqslant 2$.

5. Further Calibrations Modulo v

Section 3 showed that all of the calibrations in $\Lambda^2\mathbf{R}^{n*}$ qualify as calibrations modulo v (cf. Theorem 3.1 and [DHM, 2.2, 2.4]). The dual case $\Lambda^{n-2}\mathbf{R}^{n*}$ is of course identical. In $\Lambda^3\mathbf{R}^{6*}$, there are three nontrivial types of calibrations: forms related to the Kähler form, double forms, and the special Lagrangian form (cf. [DH] or [M2, Sect. 4]). The prototypical form of the first type, $e^*_{123} + e^*_{156}$, as a product of e^*_1 and a Kähler form, is a calibration modulo ν by Theorem 3.1 and Lemma 5.5. Theorem 4.3 shows that some double forms qualify as calibrations modulo ν. However, this section (Proposition 5.6) proves that the special Lagrangian form is not a calibration modulo ν.

In $\Lambda^3\mathbf{R}^{7*}$, there are 10 types of calibrations [HM2, Theorem 6.2]. The powerful associative calibration does not qualify modulo ν. However, the following theorem provides several which do.

5.1. THEOREM. *Suppose ϕ is a calibration in $\Lambda^3\mathbf{R}^{7*}$ of diagonal form*

$$\phi(a, b, \mu) = e_{123}^* + a_1 e_{156}^* + a_2 e_{426}^* + a_3 e_{453}^* + b_1 e_{147}^* + b_2 e_{257}^* + b_3 e_{367}^* + \mu e_{456}^*.$$

If $a_j, b_j \geqslant 0$ for all j then ϕ is a calibration modulo ν.

Proof. We must show that switching the signs on the a_j's, b_j's, and μ does not increase the comass of ϕ. For $a_j, b_j \geqslant 0$, this follows immediately from the formula of [HM2, Theorem 7.1] on the comass of diagonal forms.

5.2. EXAMPLES. Since the face $G(\phi)$ of the Grassmannian $G(3, \mathbf{R}^7)$ exposed by any diagonal calibration ϕ is identified by [HM2, Theorem 7.2], the preceding theorem gives several examples of types of faces exposed by calibrations modulo ν. New types include nonround spheres of dimensions 1, 2, and 3, including the SU_2 face described in [HM2, Sect. 8]. Consequently any sum of k 3-planes belonging to one of these faces is area-minimizing modulo ν for all $\nu \geqslant 2k$.

After the Kähler form, the most important and useful form in the theory of calibrations has been the special Lagrangian form ϕ (cf. [HL, I and III]). It is easy to check that its axis decomposition

$$\phi = e_{123}^* - e_{156}^* - e_{426}^* - e_{453}^*$$

does not exhibit ϕ as a calibration modulo ν. Indeed,

$$\left\langle \frac{e_1 + e_4}{\sqrt{2}} \wedge \frac{e_2 + e_5}{\sqrt{2}} \wedge \frac{e_3 + e_6}{\sqrt{2}}, e_{123}^* + e_{156}^* + e_{426}^* + e_{453}^* \right\rangle = 4/2\sqrt{2} > 1.$$

The following proposition implies that no decomposition works.

5.3. PROPOSITION. *There is no calibration modulo ν ϕ in $\Lambda^3\mathbf{R}^{6*}$ such that the face $G(\phi)$ contains all the special Lagrangian planes.*

Proof. The standard action of SU_3 on $\mathbf{R}^3 \oplus i\mathbf{R}^3$ leaves invariant the special Lagrangian calibration $\operatorname{Re} dz_1 \wedge dz_2 \wedge dz_3$ and hence also its face, the set $\mathscr{S}$ of special Lagrangian planes. $\mathscr{S}$ includes all planes of the form

$$\xi(\theta) = e^{i\theta_1}e_1 \wedge e^{i\theta_2}e_2 \wedge e^{i\theta_3}e_3$$

with $\theta_1 + \theta_2 + \theta_3 = 0$, because

$$\begin{aligned}\langle \xi(\theta), \operatorname{Re} dz_1 \wedge dz_2 \wedge dz_3 \rangle &= \operatorname{Re} e^{i\theta_1} e^{i\theta_2} e^{i\theta_3} \\ &= \cos(\theta_1 + \theta_2 + \theta_3).\end{aligned}$$

Suppose that there is a calibration mod v $\phi = \sum \phi_j$ such that $G(\phi) \supset \mathscr{S}$. Let $\psi = \phi_j$ for some j. We will show that $\langle \xi, \psi \rangle = 0$ for all $\xi \in \mathscr{S}$.

Using the action of SU_3, we may assume that $\psi = e_1^* \wedge e_2^* \wedge v^*$, with v perpendicular to e_1 and e_2. By Lemma 2.3 and direct computation,

$$\begin{aligned}0 &\leqslant \langle \xi(0), \psi \rangle = v \cdot e_3, \\ 0 &\leqslant \langle \xi(\pm\pi/4, \pm\pi/4, \mp\pi/2), \psi \rangle = \mp v \cdot e_6,\end{aligned}$$

so that $v \cdot e_6 = 0$; and

$$\begin{aligned}0 &\leqslant \langle \xi(\pi/3, \pi/3, -2\pi/3), \psi \rangle \\ &= -\frac{1}{8} v \cdot e_3 - \frac{\sqrt{3}}{8} v \cdot e_6 = -\frac{1}{8} v \cdot e_3,\end{aligned}$$

so that $v \cdot e_3 = 0$. Hence using the action of SU_3, we may assume that $\psi = e_1^* \wedge e_2^* \wedge (ie_1)^*$. Since the special unitary map taking (e_1, e_2, e_3) to $(e_1, -e_2, -e_3)$ maps ψ to $-\psi$ and leaves $\mathscr{S}$ invariant, the condition that $\psi \,|\, \mathscr{S} \geqslant 0$ forces $\psi \,|\, \mathscr{S} = 0$, as asserted. Therefore $\phi \,|\, \mathscr{S} = 0$, a contradiction.

5.4. LEMMA. *Let $\phi \in \Lambda^m(\mathbf{R}^n \times \mathbf{R}^l)^*$ be a calibration modulo v. Suppose $G(\phi) \cap G(m, \mathbf{R}^n) \neq \varnothing$. Then the restriction of ϕ to $\mathbf{R}^n$ is a calibration modulo v.*

Proof. Suppose $\phi = \sum \phi_j$ exhibits ϕ as a calibration mod v. Using bars to denote restriction to $\mathbf{R}^n$, we have immediately that $\bar{\phi} = \sum \bar{\phi}_j$ and for any $\xi \in G(m, \mathbf{R}^n)$, $\sigma_j = \pm 1$,

$$\left\langle \xi, \sum \sigma_j \bar{\phi}_j \right\rangle = \left\langle \xi, \sum \sigma_j \phi_j \right\rangle \leqslant 1,$$

so that $\|\sum \sigma_j \bar{\phi}_j\|^* \leqslant 1$. The hypothesis that $G(\phi) \cap G(m, \mathbf{R}^n) \neq \varnothing$ guarantees that actually $\|\bar{\phi}\|^* = 1$.

5.5. LEMMA. *Let $\phi \in \Lambda^m \mathbf{R}^{n*}$, and let $\zeta \in G(k, \mathbf{R}^l)$. Then $\phi \wedge \zeta^* \in \Lambda^{m+k}\mathbf{R}^{n+l*}$ is a calibration modulo ν if and only if ϕ is a calibration modulo ν.*

Proof. Suppose $\phi = \sum \phi_j$ exhibits ϕ as a calibration mod ν. Then $\phi \wedge \zeta^* = \sum \phi_j \wedge \zeta^*$ exhibits $\phi \wedge \zeta^*$ as a calibration mod ν. (It is easy to show that $\|\phi \wedge \zeta^*\|^* = \|\phi\|^*$.)

Conversely, suppose that $\phi \wedge \zeta^* = \sum \psi_j$ exhibits $\phi \wedge \zeta^*$ as a calibration mod ν. Then $\phi = \pm\sum \zeta \,\lrcorner\, \psi_j$ exhibits ϕ as a calibration mod ν.

The following proposition reports that the four richest calibrations of [HL, III.1.12, IV.1] do not qualify as calibrations modulo ν.

5.6. PROPOSITION. *The special Lagrangian calibration in $\Lambda^n \mathbf{R}^{2n*}$ for $n \geqslant 3$, the associative and coassociative calibrations on $\mathbf{R}^7$, and the Cayley calibration on $\mathbf{R}^8$ do not qualify as calibrations modulo ν.*

Proof. Suppose that for some $n \geqslant 3$, the special Lagrangian calibration $\operatorname{Re} dz_1 \wedge \cdots \wedge dz_n \in \Lambda^n(\mathbf{R}^n \times \mathbf{R}^n)^*$ is a calibration mod ν. By Lemma 5.4, its restriction to $\mathbf{R}^n \times \mathbf{R}^3$, $(\operatorname{Re} dz_1 \wedge dz_2 \wedge dz_3) \wedge dx_4 \wedge \cdots \wedge dx_n$, is a calibration mod ν. Then by Lemma 5.5, the special Lagrangian calibration on $\mathbf{R}^6$, $\operatorname{Re} dz_1 \wedge dz_2 \wedge dz_3$, is a calibration mod ν. This contradiction of Proposition 5.3 leads to the conclusion that the special Lagrangian form is not a calibration mod ν.

Second, suppose that the associative calibration

$$e_{234}^* - e_{278}^* - e_{638}^* - e_{674}^* - e_{265}^* - e_{375}^* - e_{485}^* \in \Lambda^3 \mathbf{R}^{8*}$$

is a calibration mod ν. By Lemma 5.4, its restriction to $e_5^\perp$, $e_{234}^* - e_{278}^* - e_{638}^* - e_{674}^*$ is a calibration mod ν. Using the isometry mapping $(e_2, e_3, e_4, e_6, e_7, e_8)$ to $(e_1, e_2, e_3, e_4, e_5, e_6)$, we deduce that the special Lagrangian form on $\mathbf{R}^6$ is a calibration mod ν, a contradiction.

Third, since the $*$-operator, an isometry of $\Lambda^3\mathbf{R}^7$ with $\Lambda^4\mathbf{R}^8$ that maps simple forms to simple forms, maps the associative calibration to the coassociative calibration, the latter is not a calibration mod ν.

Finally, since the restriction of the Cayley calibration in $\Lambda^4\mathbf{R}^{8*}$ to $\mathbf{R}^7$ is the coassociative calibration, the Cayley form is not a calibration mod ν.

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